

Maximum-Size Independent Sets and Automorphism Groups of Tensor Powers of the Even Derangement Graphs *

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Abstract

Let A_n be the alternating group of even permutations of $X := \{1, 2, \dots, n\}$ and \mathcal{E}_n the set of even derangements on X . Denote by $A\Gamma_n^q$ the tensor product of q copies of $A\Gamma_n$, where the Cayley graph $A\Gamma_n := \Gamma(A_n, \mathcal{E}_n)$ is called the even derangement graph. In this paper, we intensively investigate the properties of $A\Gamma_n^q$ including connectedness, diameter, independence number, clique number, chromatic number and the maximum-size independent sets of $A\Gamma_n^q$. By using the result on the maximum-size independent sets $A\Gamma_n^q$, we completely determine the full automorphism groups of $A\Gamma_n^q$.

Key words: Automorphism group; Cayley graph; tensor product; maximum-size independent sets; alternating group.

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1 Introduction

For a simple graph Γ , we use $V(\Gamma)$, $E(\Gamma)$ and $\text{Aut}(\Gamma)$ to denote its vertex set, edge set and full automorphism group, respectively. We denote by $N_\Gamma(v)$ the neighbourhood of a vertex v in Γ . Let G be a finite group and S a subset of G not containing the identity element 1 with $S = S^{-1}$. The *Cayley graph* $\Gamma := \Gamma(G, S)$ on G with respect to S is defined by

$$V(\Gamma) = G, E(\Gamma) = \{(g, sg) : g \in G, s \in S\}.$$

Clearly, Γ is a $|S|$ -regular and vertex-transitive graph, since $\text{Aut}(\Gamma)$ contains the right regular representation $R(G)$ of G . Moreover, Γ is connected if and only if G is generated by S .

Let S_n be the *symmetric group* and A_n the *alternating group* on $X = \{1, 2, \dots, n\}$. Let $\mathcal{D}_n := \{\sigma \in S_n : x^\sigma \neq x, \forall x \in X\}$ and $\mathcal{E}_n := \mathcal{D}_n \cap A_n$ denote the *derangements* and the *even derangements* on X respectively. Then the graph $\Gamma_n := \Gamma(S_n, \mathcal{D}_n)$ and $A\Gamma_n := \Gamma(A_n, \mathcal{E}_n)$ are called the *derangement graph* [19] and the *even derangement graph* on X respectively.

The *tensor product* $\Gamma_1 \otimes \Gamma_2$ of two graphs Γ_1 and Γ_2 is the graph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$ and edge set consisting of those pairs of vertices $(u_1, u_2), (v_1, v_2)$ where u_1 is adjacent to v_1 in Γ_1 and u_2 is adjacent to v_2 in Γ_2 . A *projection* is a homomorphism $pr_{i,n} : \Gamma^q \rightarrow \Gamma$ given by $pr_{i,n}(x_1, x_2, \dots, x_q) = x_i$ for some i , where Γ^q is the tensor product of q copies of a graph Γ . By the definition of tensor product, it is easy to see that $A\Gamma_n^q$ is the Cayley graph $\Gamma(A_n^q, \mathcal{E}_n^q)$, where A_n^q is the direct product of q copies of A_n and $\mathcal{E}_n^q := \{(\sigma_1, \sigma_2, \dots, \sigma_q) : \sigma_i \in \mathcal{E}_n, i = 1, 2, \dots, q\}$.

A family $I \subseteq S_n$ is *intersecting* if any two elements have at least one common entry. It is easy to see that an intersecting family of maximal size in S_n corresponds to a maximum-size independent set in Γ_n . In [3], Cameron and Ku showed that the only intersecting families of maximal size in S_n are the cosets of point stabilizers. In [16], Ku and Wong proved that analogous results hold for the alternating group and the direct product of symmetric groups, which equivalently shows that the structure of maximum-size independent sets of $A\Gamma_n$ is as follows:

Proposition 1.1 (*Theorem 1.2 in [16]*) *All the maximum-size independent sets of $A\Gamma_n$ ($n \geq 5$) are $B_{i,j} = \{\sigma \in A_n : i^\sigma = j\}$, $i, j = 1, 2, \dots, n$. In particular, each $|B_{i,j}| = \frac{(n-1)!}{2}$.*

In this paper, we prove that the result analogous to [3] holds for the direct product of the alternating groups, which can be equivalently stated as follows:

Theorem 1.2 *All the maximum-size independent sets of AI_n^q ($q \geq 1, n \geq 5$) are*

$$B_{i,j}^{(k)} = \{(\sigma_1, \sigma_2, \dots, \sigma_q) \in A_n^q : i^{\sigma_k} = j\}, i, j = 1, 2, \dots, n; k = 1, 2, \dots, q.$$

In particular, the independence number of AI_n^q is

$$|B_{i,j}^{(k)}| = \frac{(n-1)!n^{q-1}}{2^q}.$$

Remark. Generally speaking, for a graph Γ , all maximum-size independent sets of Γ^q are not necessarily preimages of maximum-size independent sets of Γ under projections (see [15, 18]). Theorem 1.2 shows that all maximum-size independent sets of AI_n^q are preimages of maximum-size independent sets of AI_n under projections.

Many researchers (see [3, 4, 5, 17, 19, 20]) have studied the properties of Γ_n , such as the clique number, the chromatic number, the independence number, maximum-size independent sets and so on. Motivated by the nice structures of Γ_n , here we show that AI_n^q have the similar nice structures. For example, we obtain that the diameter $D(AI_n^q) = 2$, the clique number $\omega(AI_n^q) = n$ and the chromatic number $\chi(AI_n^q) = n$.

Cayley graphs are of general interest in the field of Algebraic Graph Theory due to their good properties, especially their high symmetry. One difficult problem in Algebraic Graph Theory is to determine the automorphism groups of Cayley graphs. Although there are some nice results on the automorphism groups of Cayley graphs (see [6, 7, 8, 10, 13, 23, 24, 25]), we still lack enough understanding on them. In this paper, we completely determine the automorphism groups of AI_n^q , which in fact gives a kind of method on the computation of automorphism group of Cayley graph by using the characterization of the maximum-size independent sets. Another main result of this paper is as follows:

Theorem 1.3 *Define the mapping $\varphi_k : A_n^q \rightarrow A_n^q$ as $(\sigma_1, \dots, \sigma_{k-1}, \sigma_k, \sigma_{k+1}, \dots, \sigma_q)^{\varphi_k} = (\sigma_1, \dots, \sigma_{k-1}, \sigma_k^{-1}, \sigma_{k+1}, \dots, \sigma_q)$ for $k = 1, 2, \dots, q$. For $q \geq 1$ and $n \geq 5$,*

$$\text{Aut}(AI_n^q) = (R(A_n^q) \rtimes (\text{Inn}(S_n) \wr S_q)) \rtimes Z_2^q,$$

where $\text{Inn}(S_n) (\cong S_n)$ is the inner automorphism group of S_n , $Z_2^q = \langle \varphi_1 \rangle \times \langle \varphi_2 \rangle \times \dots \times \langle \varphi_q \rangle$ and $\text{Inn}(S_n) \wr S_q$ denotes the wreath product of $\text{Inn}(S_n)$ and S_q .

Remark. Sanders and George [21] showed that for a graph Γ , $\text{Aut}(\Gamma^2) \geq \text{Aut}(\Gamma) \wr S_2$, where \wr denotes the wreath product, however, the equality cannot hold in most situations. Theorem 1.3 implies that $\text{Aut}(AI_n^q) = \text{Aut}(AI_n) \wr S_q$.

The rest part of this paper is organized as follows. In Section 2, we give the connectedness and diameter of $A\Gamma_n^q$. In Section 3, we determine the independence number and the structure of maximum-size independent sets of $A\Gamma_n^q$, as its corollary, we obtain the clique number and chromatic number of $A\Gamma_n^q$. In section 4, we completely determine the full automorphism groups of $A\Gamma_n^q$.

2 The connectedness and diameter

In this section, we give the connectedness and diameter of $A\Gamma_n^q$.

For a group G , we denote the automorphism group and the inner automorphism group of G by $\text{Aut}(G)$ and $\text{Inn}(G)$, respectively. Next we need the following known result:

Proposition 2.1 [22] [III, (2.17) – (2.20)] *If $n \geq 2$ and $n \neq 6$, then $\text{Aut}(A_n) = \text{Inn}(S_n)$. If $n = 6$, then $|\text{Aut}(A_6) : \text{Inn}(S_6)| = 2$, and for each $\alpha \in \text{Aut}(A_6) \setminus \text{Inn}(S_6)$, α maps a 3-cycle to a product of two disjoint 3-cycles.*

Lemma 2.2 *If $n \geq 5$, then the even derangement graph $A\Gamma_n$ is connected.*

Proof. By Theorem 2.8 of page 293 in [22], the alternating group A_n ($n \geq 5$) is generated by the totality of 3-cycles. Clearly $(1\ 2\ 3) = (1\ 2 \cdots n)^2 \cdot (n\ n-1 \cdots 1)^2(1\ 2\ 3)$ and $(1\ 2 \cdots n)^2, (n\ n-1 \cdots 1)^2(1\ 2\ 3) \in \mathcal{E}_n$ by $n \geq 5$.

For any 3-cycle $(i\ j\ k)$, there exists a $\phi \in \text{Inn}(S_n)$ such that $(1\ 2\ 3)^\phi = (i\ j\ k)$. By Proposition 2.1, we have $\text{Aut}(A_n, \mathcal{E}_n) = \{\phi \in \text{Aut}(A_n) : \mathcal{E}_n^\phi = \mathcal{E}_n\} = \text{Inn}(S_n)$. Thus

$$(i\ j\ k) = (1\ 2\ 3)^\phi = [(1\ 2 \cdots n)^2]^\phi \cdot [(n\ n-1 \cdots 1)^2(1\ 2\ 3)]^\phi$$

and

$$[(1\ 2 \cdots n)^2]^\phi, [(n\ n-1 \cdots 1)^2(1\ 2\ 3)]^\phi \in \mathcal{E}_n.$$

So the alternating group A_n ($n \geq 5$) is generated by \mathcal{E}_n , which implies that $A\Gamma_n$ is connected. ■

Remark. If $n = 3$, clearly $A_3 = \langle \mathcal{E}_3 \rangle$, so $A\Gamma_3$ is connected. If $n = 4$, then $A_4 \neq \langle \mathcal{E}_4 \rangle = \{1, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$, so $A\Gamma_4$ is not connected.

Lemma 2.3 ^[12] (i) *The tensor product of two connected graphs is bipartite if and only if at least one of them is bipartite.*

(ii) *The tensor product of two connected graphs is disconnected if and only if both factors are bipartite.*

Theorem 2.4 $A\Gamma_n^q$ is connected and non-bipartite for any $q \geq 1$ and $n \geq 5$.

Proof. Since $A\Gamma_n^q = \underbrace{A\Gamma_n \otimes \cdots \otimes A\Gamma_n}_q$ and $A\Gamma_n$ is connected and non-bipartite for $n \geq 5$ by Lemma 2.2, the assertion holds by Lemma 2.3. ■

Lemma 2.5 For any $g_1, g_2 \in A_n$ ($n \geq 5$), there exists a $g \in A_n$ such that $g \in N_{A\Gamma_n}(g_1) \cap N_{A\Gamma_n}(g_2)$.

Proof. If $n = 5$, we have

$$\begin{aligned} (a_1, a_2, a_3, a_4, a_5) &= (a_1, a_4, a_2, a_5, a_3)^2, \\ (a_1, a_2)(a_3, a_4) &= (a_5, a_1, a_3, a_2, a_4)(a_1, a_5, a_3, a_2, a_4), \\ (a_1, a_2, a_3) &= (a_1, a_5, a_3, a_4, a_2)(a_1, a_3, a_5, a_2, a_4), \\ 1 &= (a_1, a_2, a_3, a_4, a_5)(a_5, a_4, a_3, a_2, a_1), \end{aligned}$$

that is, for any $x \in A_5$, there exist $s_1, s_2 \in \mathcal{E}_5$ such that $x = s_1 s_2$. Now for $x = g_1 g_2^{-1}$, we have $g_1 g_2^{-1} = s_1 s_2$, $s_1, s_2 \in \mathcal{E}_5$, i.e. $g_1 = s_1 s_2 g_2$. Set $g := s_2 g_2$. Clearly $g \in N_{A\Gamma_5}(g_1) \cap N_{A\Gamma_5}(g_2)$.

If $n \geq 6$, by proposition 6 in [3], for any $g_1, g_2 \in A_n$, there exists a $g \in S_n$ such that $g \in N_{\Gamma_n}(g_1) \cap N_{\Gamma_n}(g_2)$. That is, there exist $s_1, s_2 \in \mathcal{D}_n$ such that $g = s_1 g_1 = s_2 g_2$. If $g \in A_n$, then $s_1, s_2 \in \mathcal{E}_n$, so $g \in N_{A\Gamma_n}(g_1) \cap N_{A\Gamma_n}(g_2)$ and the assertion holds. If $g \in S_n \setminus A_n$, then $s_1, s_2 \in \mathcal{D}_n \setminus \mathcal{E}_n$. For any $i \in X = \{1, 2, \dots, n\}$, select a $j \in \{i, i^{s_1}, i^{s_2}, i^{s_1^{-1}}, i^{s_2^{-1}}\} \neq \emptyset$ ($n \geq 6$). Set

$$g' := (i j)g, \quad s'_1 := (i j)s_1, \quad s'_2 := (i j)s_2.$$

Thus $g' = s'_1 g_1 = s'_2 g_2$ and $s'_1, s'_2 \in \mathcal{E}_n$ by $j \in X \setminus \{i, i^{s_1}, i^{s_2}, i^{s_1^{-1}}, i^{s_2^{-1}}\}$. Hence $g' \in N_{A\Gamma_n}(g_1) \cap N_{A\Gamma_n}(g_2)$ and the assertion holds. ■

Theorem 2.6 If $n \geq 5$, then $\text{diam}(A\Gamma_n^q) = 2$, where $\text{diam}(A\Gamma_n^q)$ is the diameter of $A\Gamma_n^q$.

Proof. For any $(\sigma_1, \sigma_2, \dots, \sigma_q), (\tau_1, \tau_2, \dots, \tau_q) \in A_n^q$, by Lemma 2.5, there exist $\varsigma_i \in A_n$ ($i = 1, 2, \dots, q$) such that $\varsigma_i \in N_{A\Gamma_n}(\sigma_i) \cap N_{A\Gamma_n}(\tau_i)$. So there exists a $(\varsigma_1, \varsigma_2, \dots, \varsigma_q) \in A_n^q$ such that

$$(\varsigma_1, \varsigma_2, \dots, \varsigma_q) \in N_{A\Gamma_n^q}((\sigma_1, \sigma_2, \dots, \sigma_q)) \cap N_{A\Gamma_n^q}((\tau_1, \tau_2, \dots, \tau_q)),$$

which implies that any two vertices in $A\Gamma_n^q$ have at least a common neighbourhood. Hence $\text{diam}(A\Gamma_n^q) = 2$. ■

3 The stucture of maximum-size independent sets

In this section we characterize the structure of maximum-size independent sets of $A\Gamma_n^q$ for $q \geq 1$, which is a generalization of Theorem 1.2 in [16]. First we give the independence number of $A\Gamma_n^q$ as follows:

Lemma 3.1 *For any $q \geq 1$, $n \geq 5$, the independence number of $A\Gamma_n^q$ is given by*

$$\alpha(A\Gamma_n^q) = \frac{(n-1)!n^{q-1}}{2^q}.$$

Proof. By Proposition 1.3 in [2], we have

$$\frac{\alpha(A\Gamma_n^q)}{|A_n^q|} = \frac{\alpha(A\Gamma_n)}{|A_n|} \Rightarrow \alpha(A\Gamma_n^q) = \frac{\alpha(A\Gamma_n) \cdot |A_n^q|}{|A_n|}.$$

Then by Proposition 1.1, we obtain

$$\alpha(A\Gamma_n^q) = \frac{\frac{(n-1)!}{2} \cdot \left(\frac{n!}{2}\right)^q}{\frac{n!}{2}} = \frac{(n-1)!n^{q-1}}{2^q}.$$

Thus the assertion holds. ■

For any two graphs H_1 and H_2 , a map ϕ from $V(H_1)$ to $V(H_2)$ is *homomorphism* if $\{u^\phi, v^\phi\} \in E(H_2)$ whenever $\{u, v\} \in E(H_1)$, i.e. ϕ is a edge-preserving map. Next we need the following fundamental result of Albertson and Collins [1] which is also called 'No-Homomorphism Lemma'.

Lemma 3.2 [1] *Let H_1 and H_2 be graphs such that H_2 is vertex transitive and there exists a homomorphism $\phi : V(H_1) \rightarrow V(H_2)$. Then*

$$\frac{\alpha(H_1)}{|V(H_1)|} \geq \frac{\alpha(H_2)}{|V(H_2)|} \quad (1)$$

Furthermore, if equality holds in (1), then for any independent set I of cardinality $\alpha(H_2)$ in H_2 , $I^{\phi^{-1}}$ is an independent set of cardinality $\alpha(H_1)$ in H_1 .

Lemma 3.3 *Let $H = (V_1, V_2, E)$ be a d -regular bipartite graph whose partition has the parts V_1 and V_2 with $|V_1| = |V_2|$. If H is connected, then $|S| < |N_H(S)|$ for any $S \subsetneq V_1$, where $N_H(S)$ is the neighborhood of S in H .*

Proof. Let $T = N_H(S)$ and $E(S, T) = \{(s, t) \in E : s \in S, t \in T\}$. Then

$$d|S| = |E(S, T)| \leq |E(V_1, T)| = d|T|.$$

If $|S| = |T|$, then $|E(S, T)| = |E(V_1, T)|$, i.e. any vertex $u \in S \cup T$ is not adjacent to any vertex $v \notin S \cup T$, which contradicts the connectedness of H . Thus $|S| < |T| = |N_H(S)|$. ■

Lemma 3.4 *All the maximum-size independent sets of AI_n^2 ($n \geq 7$) are*

$$B_{i,j}^{(k)} = \{(g_1, g_2) \in A_n^2 : i^{g_k} = j\}, i, j = 1, 2, \dots, n; k = 1, 2.$$

Proof. Set $\mathcal{B} = \{B_{i,j}^{(k)} : i, j = 1, 2, \dots, n; k = 1, 2\}$. Clearly $|B_{i,j}^{(k)}| = \frac{(n-1)!n!}{4}$, which is equal to $\alpha(AI_n^2)$ by Lemma 3.1. That is, $B_{i,j}^{(k)}$ is a maximum-size independent set of AI_n^2 . Next for any maximum-size independent set I of AI_n^2 , it suffices to show that $I \in \mathcal{B}$.

Define a homomorphism ϕ from AI_n to AI_n^2 as $g^\phi = (g, g)$. Without loss of generality, we may assume that the identity $Id = (id, id) \in I$. By Proposition 1.1, Lemmas 3.1 and 3.2, $I^{\phi^{-1}}$ is a maximum independent set of AI_n . So $I^{\phi^{-1}} = \{g \in A_n : i_0^g = j_0\}$ for some i_0, j_0 by Proposition 1.1. Since $id = (Id)^{\phi^{-1}} \in I^{\phi^{-1}}$, $I^{\phi^{-1}} = \{g \in A_n : i_0^g = i_0\}$ for some i_0 . Therefore $I \supseteq I_0 := (I^{\phi^{-1}})^\phi = \{(g, g) \in A_n^2 : i_0^g = i_0\}$. Next we shall show that $I \in \mathcal{B}$ by the following four Claims:

Claim 1. For any $(g_1, g_2) \in I$, either $i_0^{g_1} = i_0$ or $i_0^{g_2} = i_0$.

Suppose on the contrary that $i_0^{g_1} \neq i_0$ and $i_0^{g_2} \neq i_0$. By Lemma 2.5, there exists a $g \in A_n$ such that $g \in N_{AI_n}(g_1) \cap N_{AI_n}(g_2)$. That is, there exist $s_1, s_2 \in \mathcal{E}_n$ such that $g = s_1 g_1 = s_2 g_2$.

If $i_0^g = i_0$, then $(g, g) \in I_0 \subseteq I$ and $\{(g, g), (g_1, g_2)\} \in E(AI_n^2)$, which contradicts the fact that $(g_1, g_2) \in I$ and I is an independent set.

If $i_0^g \neq i_0$, select a $j \in X \setminus \{i_0, i_0^{g_1^{-1}}, i_0^{s_1}, i_0^{s_2}, i_0^{g_1^{-1}s_1^{-1}}, i_0^{g_1^{-1}s_2^{-1}}\} \neq \emptyset$ ($n \geq 7$). Set

$$g' = (i_0, i_0^{g_1^{-1}}, j)g, s'_1 = (i_0, i_0^{g_1^{-1}}, j)s_1, s'_2 = (i_0, i_0^{g_1^{-1}}, j)s_2.$$

Thus $i_0^{g'} = i_0$, $g' = s'_1 g_1 = s'_2 g_2$ and $s'_1, s'_2 \in \mathcal{E}_n$ by $j \in X \setminus \{i_0, i_0^{g_1^{-1}}, i_0^{s_1}, i_0^{s_2}, i_0^{g_1^{-1}s_1^{-1}}, i_0^{g_1^{-1}s_2^{-1}}\}$. So $(g', g') \in I_0 \subseteq I$ and $\{(g', g'), (g_1, g_2)\} \in E(AI_n^2)$, which as above yields a contradiction.

Hence Claim 1 holds.

Set

$$J_0 = \{(g_1, g_2) \in A_n^2 : i_0^{g_1} = i_0 \text{ and } i_0^{g_2} = i_0\},$$

$$J_1 = \{(g_1, g_2) \in A_n^2 : i_0^{g_1} = i_0 \text{ and } i_0^{g_2} \neq i_0\},$$

$$J_2 = \{(g_1, g_2) \in A_n^2 : i_0^{g_1} \neq i_0 \text{ and } i_0^{g_2} = i_0\}.$$

$$\text{Clearly } |J_0| = \frac{(n-1)!^2}{4}, |J_1| = |J_2| = \frac{(n-1)(n-1)!^2}{4}.$$

Claim 2. $AI_n^2[J_1 \cup J_2]$ is connected, where $AI_n^2[J_1 \cup J_2]$ denote the induced subgraph of AI_n^2 by $J_1 \cup J_2$.

For any $(\sigma_1, \sigma_2), (\tau_1, \tau_2) \in J_1$, clearly they are not adjacent in AI_n^2 . By Theorem 2.6, there exists a $(\varsigma_1, \varsigma_2) \in A_n^2$ such that $\{(\sigma_1, \sigma_2), (\varsigma_1, \varsigma_2)\}$ and $\{(\tau_1, \tau_2), (\varsigma_1, \varsigma_2)\} \in E(AI_n^2)$. That is, there exist $s_1, s_2, t_1, t_2 \in \mathcal{E}_n$ such that $\varsigma_1 = s_1\sigma_1 = t_1\tau_1$, $\varsigma_2 = s_2\sigma_2 = t_2\tau_2$. Clearly $i_0^{\varsigma_1} = i_0^{s_1\sigma_1} \neq i_0$.

If $i_0^{\varsigma_2} = i_0$, then $(\varsigma_1, \varsigma_2) \in J_2$.

If $i_0^{\varsigma_2} \neq i_0$, then select a $j \in X \setminus \{i_0, i_0^{s_2^{-1}}, i_0^{s_2}, i_0^{t_2}, i_0^{s_2^{-1}s_2^{-1}}, i_0^{s_2^{-1}t_2^{-1}}\} \neq \emptyset$ ($n \geq 7$). Set

$$\varsigma'_2 = (i_0, i_0^{s_2^{-1}}, j)\varsigma_2, s'_2 = (i_0, i_0^{s_2^{-1}}, j)s_2, t'_2 = (i_0, i_0^{s_2^{-1}}, j)t_2.$$

Thus and $i_0^{\varsigma'_2} = i_0$, $\varsigma'_2 = s'_2\sigma_2 = t'_2\tau_2$ and $s'_2, t'_2 \in \mathcal{E}_n$ by $j \in X \setminus \{i_0, i_0^{s_2^{-1}}, i_0^{s_2}, i_0^{t_2}, i_0^{s_2^{-1}s_2^{-1}}, i_0^{s_2^{-1}t_2^{-1}}\}$. So $(\varsigma_1, \varsigma'_2) \in J_2$ and $\{(\sigma_1, \sigma_2), (\varsigma_1, \varsigma'_2)\}, \{(\tau_1, \tau_2), (\varsigma_1, \varsigma'_2)\} \in E(AI_n^2[J_1 \cup J_2])$.

Similarly, for any $(\sigma_1, \sigma_2), (\tau_1, \tau_2) \in J_2$, their exists $(\varsigma_1, \varsigma_2) \in J_1$ such that $\{(\sigma_1, \sigma_2), (\varsigma_1, \varsigma_2)\}, \{(\tau_1, \tau_2), (\varsigma_1, \varsigma_2)\} \in E(AI_n^2[J_1 \cup J_2])$.

Hence Claim 2 holds.

Claim 3. Either $I \cap J_1 = \emptyset$ or $I \cap J_2 = \emptyset$.

Suppose on the contrary that $I \cap J_1 \neq \emptyset$ and $I \cap J_2 \neq \emptyset$, consider the following two possible cases:

Case 1. $I \cap J_1 = J_1$ or $I \cap J_2 = J_2$.

Since $I \cap (J_1 \cup J_2)$ is an independent set, this case cannot happen.

Case 2. $I \cap J_1 \subsetneq J_1$ and $I \cap J_2 \subsetneq J_2$.

It is easy to see that $AI_n^2[J_1 \cup J_2]$ is a regular bipartite graph whose partition has the parts J_1 and J_2 with $|J_1| = |J_2|$. By Claim 2 and Lemma 3.3, we have

$$|I \cap J_1| < |N_{AI_n^2[J_1 \cup J_2]}(I \cap J_1)|.$$

Since $I \cap (J_1 \cup J_2)$ is an independent set, we have

$$\begin{aligned} I \cap J_2 &\subseteq J_2 \setminus N_{AI_n^2[J_1 \cup J_2]}(I \cap J_1) \\ \Rightarrow |N_{AI_n^2[J_1 \cup J_2]}(I \cap J_1)| + |I \cap J_2| &\leq |J_2| \\ \Rightarrow |I \cap J_1| + |I \cap J_2| &< |J_2| \end{aligned}$$

By Claim 1, $I = \bigcup_{i=0}^2 (I \cap J_i)$. Since J_i ($i = 0, 1, 2$) are pairwise disjoint, we have

$$\begin{aligned} |I| &= |I \cap J_0| + |I \cap J_1| + |I \cap J_2| \\ &< |J_0| + |J_2| \\ &= \frac{(n-1)!^2}{4} + \frac{(n-1)(n-1)!^2}{4} \\ &= \frac{(n-1)!n!}{4} \end{aligned}$$

which is a contradiction, since $|I| = \frac{(n-1)!n!}{4}$ by Lemma 3.1. Hence Claim 3 holds.

Claim 4. Either $I = J_0 \cup J_1$ or $I = J_0 \cup J_2$.

By Claim 3, either $I \cap J_1 = \emptyset$ or $I \cap J_2 = \emptyset$. If $I \cap J_1 = \emptyset$, then we have

$$\begin{aligned} \frac{(n-1)!n!}{4} = |I| &= |I \cap J_0| + |I \cap J_2| \leq |J_0| + |J_2| = \frac{(n-1)!n!}{4} \\ \Rightarrow I \cap J_0 &= J_0, I \cap J_2 = J_2. \end{aligned}$$

So $I = \bigcup_{i=0}^2 (I \cap J_i) = J_0 \cup J_2$.

Similarly, if $I \cap J_2 = \emptyset$, then $I = J_0 \cup J_1$. Hence Claim 4 holds.

By Claim 4, we have $I \in B$, which conclude the proof. ■

Lemma 3.5 [2] *Let Γ be a connected d -regular graph on n vertices and let $d = \mu_1 \geq \mu_2 \geq \dots \mu_n$ be the eigenvalues of the adjacency matrix of Γ . If*

$$\frac{\alpha(\Gamma)}{n} = \frac{-\mu_n}{d - \mu_n},$$

then for every integer $q \geq 1$,

$$\frac{\alpha(\Gamma^q)}{n^q} = \frac{-\mu_n}{d - \mu_n}.$$

Moreover, if Γ is also non-bipartite, and if I is an independent set of size $\frac{-\mu_n}{d - \mu_n} n^q$ in Γ^q , then there exists a coordinate $i \in \{1, 2, \dots, q\}$ and a maximum-size independent set J in Γ , such that

$$I = \{(v_1, \dots, v_q \in V(\Gamma^q) : v \in J\}.$$

Lemma 3.6 [15] *Let Γ be a connected, non-bipartite vertex-transitive graph. Suppose that the only independent sets of maximal cardinality in H^2 are the preimages of the independent sets of maximal cardinality in Γ under projections. Then the same holds for all powers of Γ .*

Proof. (of Theorem 1.2). For $n = 5, 6$, it is easy to see that $A\Gamma_n$ is connected, non-bipartite and $e(n)$ -regular graph with $e(5) = 24$ and $e(6) = 130$. Moreover, a **Matlab** computation shows that the least eigenvalue of the adjacency matrix of $A\Gamma_5$ and $A\Gamma_6$ are -6 and -26 , respectively. Thus the assertion holds by Lemmas 3.1 and 3.5.

For $n \geq 7$, combining Proposition 1.1, Lemma 3.4 and 3.6, the assertion holds. ■

Corollary 3.7 *Let $\omega(A\Gamma_n^q)$ and $\chi(A\Gamma_n^q)$ denote the clique number and chromatic number of $A\Gamma_n^q$ ($n \geq 5$). Then we have*

$$\omega(A\Gamma_n^q) = \chi(A\Gamma_n^q) = n.$$

Proof. By [16], we have $\omega(A\Gamma_n) = n$. Let $\{\sigma_1, \sigma_2, \dots, \sigma_n\}$ be a clique of $A\Gamma_n$. Then clearly $\{(\sigma_1, \sigma_1, \dots, \sigma_1), (\sigma_2, \sigma_2, \dots, \sigma_2), \dots, (\sigma_n, \sigma_n, \dots, \sigma_n)\}$ is a clique of $A\Gamma_n^q$. So we have $\omega(A\Gamma_n^q) \geq n$. On the other hand, by Theorem 1.2, we know that the independence number $\alpha(A\Gamma_n^q) = \frac{(n-1)!n^{q-1}}{2^q}$. By Corollary 4 in [3], we have $\omega(A\Gamma_n^q)\alpha(A\Gamma_n^q) \leq |V(A\Gamma_n^q)|$, that is $\omega(A\Gamma_n^q) \cdot \frac{(n-1)!n^{q-1}}{2^q} \leq \frac{n!q}{2^q}$, so $\omega(A\Gamma_n^q) \leq n$. Thus $\omega(A\Gamma_n^q) = n$.

In addition, by Corollary 6.1.3 in [9], for any Cayley graph $\Gamma := \Gamma(G, S)$, if S is closed under conjugation and $\alpha(\Gamma)\omega(\Gamma) = |V(\Gamma)|$, then $\chi(\Gamma) = \omega(\Gamma)$. Note that for $A\Gamma_n^q = \Gamma(A_n^q, \mathcal{E}_n^q)$, \mathcal{E}_n^q is closed under conjugation and $\alpha(A\Gamma_n^q)\omega(A\Gamma_n^q) = |V(A\Gamma_n^q)|$. Hence $\chi(A\Gamma_n^q) = \omega(A\Gamma_n^q) = n$. ■

4 The automorphism group of $A\Gamma_n^q$

In this section, we completely determine the full automorphism group of $A\Gamma_n^q$ ($n \geq 5$). First we introduce some definitions. Let $\text{Sym}(\Omega)$ denote the set of all permutations of a set Ω . A *permutation representation* of a group G is a homomorphism from G into $\text{Sym}(\Omega)$ for some set Ω . A permutation representation is also referred to as an action of G on the set Ω , in which case we say that G acts on Ω . Furthermore, if $\{g \in G : x^g = x, \forall x \in \Omega\} = 1$, we say the action of G on Ω is *faithful*, or G acts *faithfully* on Ω .

Next we need the following known results:

Proposition 4.1 [14] *Let $G^q = G \times G \times \dots \times G$ be the external direct product of q copies of the nontrivial group G . If G has the following properties:*

- (i) *the center $Z(G)$ of G is trivial;*
- (ii) *G cannot be decomposed as a nontrivial direct product.*

Then $\text{Aut}(G^q) = \text{Aut}(G) \wr S_q$.

Proposition 4.2 [11] *Let $N_{\text{Aut}(\Gamma(G,S))}(R(G))$ be the normalizer of $R(G)$ in $\text{Aut}(\Gamma(G,S))$. Then*

$$N_{\text{Aut}(\Gamma(G,S))}(R(G)) = R(G) \rtimes \text{Aut}(G, S) \leq \text{Aut}(\Gamma(G, S)),$$

where $\text{Aut}(G, S) = \{\phi \in \text{Aut}(G) : S^\phi = S\}$.

Lemma 4.3 *Define the mapping $\varphi_k : A_n^q \rightarrow A_n^q$ as $(\sigma_1, \dots, \sigma_{k-1}, \sigma_k, \sigma_{k+1}, \dots, \sigma_q)^{\varphi_k} = (\sigma_1, \dots, \sigma_{k-1}, \sigma_k^{-1}, \sigma_{k+1}, \dots, \sigma_q)$ for $k = 1, 2, \dots, q$. For $n \geq 5$,*

$$(R(A_n^q) \rtimes (\text{Inn}(S_n) \wr S_q)) \rtimes Z_2^q \leq \text{Aut}(A\Gamma_n^q),$$

where $\text{Inn}(S_n) \cong S_n$ and $Z_2^q = \langle \varphi_1 \rangle \times \langle \varphi_2 \rangle \times \dots \times \langle \varphi_q \rangle$. In particular, $|\text{Aut}(A\Gamma_n^q)| \geq |(R(A_n^q) \rtimes (\text{Inn}(S_n) \wr S_q)) \rtimes Z_2^q| = q!n!^{2q}$.

Proof. By Proposition 2.1 and 4.1, we have

$$\begin{aligned} \text{Aut}(A_n^q, \mathcal{E}_n^q) &= \{\phi \in \text{Aut}(A_n^q) : (\mathcal{E}_n^q)^\phi = \mathcal{E}_n^q\} \\ &= \{\phi \in \text{Aut}(A_n) \wr S_q : (\mathcal{E}_n^q)^\phi = \mathcal{E}_n^q\} \\ &= \text{Inn}(S_n) \wr S_q. \end{aligned}$$

Using Proposition 4.2, we obtain $R(A_n^q) \rtimes (\text{Inn}(S_n) \wr S_q) \leq \text{Aut}(A\Gamma_n^q)$.

Next we show that φ_k is an automorphism of $A\Gamma_n^q$.

$$\begin{aligned} &\{(\sigma_1, \dots, \sigma_k, \dots, \sigma_q), (\tau_1, \dots, \tau_k, \dots, \tau_q)\} \in E(A\Gamma_n^q) \\ \Leftrightarrow &\forall i \in \{1, 2, \dots, n\}, \forall k \in \{1, 2, \dots, q\}, i^{\sigma_k} \neq i^{\tau_k} \\ \Leftrightarrow &\forall i \in \{1, 2, \dots, n\}, \forall k \in \{1, 2, \dots, q\}, (i^{\sigma_k^{-1}})^{\sigma_k} \neq (i^{\sigma_k^{-1}})^{\tau_k} \\ \Leftrightarrow &\forall i \in \{1, 2, \dots, n\}, \forall k \in \{1, 2, \dots, q\}, i \neq i^{\sigma_k^{-1}\tau_k} \\ \Leftrightarrow &\forall i \in \{1, 2, \dots, n\}, \forall k \in \{1, 2, \dots, q\}, i^{\tau_k^{-1}} \neq i^{\sigma_k^{-1}} \\ \Leftrightarrow &\{(\sigma_1, \dots, \sigma_k^{-1}, \dots, \sigma_q), (\tau_1, \dots, \tau_k^{-1}, \dots, \tau_q)\} \in E(A\Gamma_n^q) \\ \Leftrightarrow &\{(\sigma_1, \dots, \sigma_k, \dots, \sigma_q)^{\varphi_k}, (\tau_1, \dots, \tau_k, \dots, \tau_q)^{\varphi_k}\} \in E(A\Gamma_n^q). \end{aligned}$$

It is easy to see that $\varphi_k \notin R(A_n^q)$ and $\varphi_k \notin \text{Inn}(S_n) \wr S_q$. Hence

$$(R(A_n^q) \rtimes (\text{Inn}(S_n) \wr S_q)) \rtimes Z_2^q \leq \text{Aut}(A\Gamma_n^q),$$

where $Z_2^q = \langle \varphi_1 \rangle \times \langle \varphi_2 \rangle \times \dots \times \langle \varphi_q \rangle$. The assertion holds. ■

Lemma 4.4 Let $\mathcal{B} = \{B_{i,j}^{(k)}, i, j = 1, 2, \dots, n; k = 1, 2, \dots, q\}$, where $B_{i,j}^{(k)} = \{(\sigma_1, \sigma_2, \dots, \sigma_q) \in A_n^q : i^{\sigma_k} = j\}$. Then the action of $\text{Aut}(A\Gamma_n^q)$ on \mathcal{B} can be induced by the natural action of $\text{Aut}(A\Gamma_n^q)$ on A_n^q , and is faithful. Furthermore, any $\phi \in \text{Aut}(A\Gamma_n^q)$ is a permutation of \mathcal{B} .

Proof. Obviously, any $\phi \in \text{Aut}(A\Gamma_n^q)$ maps a maximum-size independent set of $A\Gamma_n^q$ to a maximum-size independent set of $A\Gamma_n^q$. So by Theorem 1.2, for any $B_{i,j}^{(k)} \in \mathcal{B}$ and $\phi \in \text{Aut}(A\Gamma_n^q)$, we have $B_{i,j}^{(k)\phi} \in \mathcal{B}$.

Next we show that if $\phi \in \text{Aut}(A\Gamma_n^q)$ satisfies $B_{i,j}^{(k)\phi} = B_{i,j}^{(k)}$ for each $B_{i,j}^{(k)} \in \mathcal{B}$, then ϕ is the identity map. In fact, clearly,

$$\forall (\sigma_1, \sigma_2, \dots, \sigma_q) \in A_n^q, \{(\sigma_1, \sigma_2, \dots, \sigma_q)\} = \bigcap_{k=1}^q \bigcap_{i=1}^n B_{i,i^{\sigma_k}}^{(k)}.$$

So

$$\begin{aligned} \{(\sigma_1, \sigma_2, \dots, \sigma_q)^\phi\} &= \left(\bigcap_{k=1}^q \bigcap_{i=1}^n B_{i,i^{\sigma_k}}^{(k)} \right)^\phi \\ &\subseteq \bigcap_{k=1}^q \bigcap_{i=1}^n B_{i,i^{\sigma_k}}^{(k)\phi} \\ &= \bigcap_{k=1}^q \bigcap_{i=1}^n B_{i,i^{\sigma_k}}^{(k)} \\ &= \{(\sigma_1, \sigma_2, \dots, \sigma_q)\}. \end{aligned}$$

Thus ϕ is the identity map.

For any $B_{i,j}^{(k)}, B_{i',j'}^{(k')} \in \mathcal{B}$ and $\phi \in \text{Aut}(A\Gamma_n^q)$, we have

$$\begin{aligned} B_{i,j}^{(k)} \neq B_{i',j'}^{(k')} &\Leftrightarrow |B_{i,j}^{(k)} \cup B_{i',j'}^{(k')}| > \frac{(n-1)!n^{q-1}}{2^q} \\ &\Leftrightarrow |(B_{i,j}^{(k)} \cup B_{i',j'}^{(k')})^\phi| > \frac{(n-1)!n^{q-1}}{2^q} \\ &\Leftrightarrow |B_{i,j}^{(k)\phi} \cup B_{i',j'}^{(k')\phi}| > \frac{(n-1)!n^{q-1}}{2^q} \\ &\Leftrightarrow B_{i,j}^{(k)\phi} \neq B_{i',j'}^{(k')\phi}. \end{aligned}$$

Thus ϕ is a permutation of \mathcal{B} . ■

Lemma 4.5 $B_{i,j}^{(k)} \cap B_{i',j'}^{(k')} = \emptyset$ if and only if $k = k'$ and exactly one of $i = i'$ and $j = j'$ holds.

Proof. If $k = k'$ and exactly one of $i = i'$ and $j = j'$ holds, then $B_{i,j}^{(k)} \cap B_{i',j'}^{(k')} = \emptyset$.

If $k \neq k'$, then $|B_{i,j}^{(k)} \cap B_{i',j'}^{(k')}| = | \{ (\sigma_1, \sigma_2, \dots, \sigma_q) \in A_n^q : i^{\sigma_k} = j, i'^{\sigma_{k'}} = j' \} | = \frac{(n-1)!n^{!q-2}}{2^q}$.

If $k = k'$, $i = i'$ and $j = j'$, then $B_{i,j}^{(k)} = B_{i',j'}^{(k')}$, so $B_{i,j}^{(k)} \cap B_{i',j'}^{(k')} \neq \emptyset$.

If $k = k'$, $i \neq i'$ and $j \neq j'$, then

$$|B_{i,j}^{(k)} \cap B_{i',j'}^{(k')}| = | \{ (\sigma_1, \sigma_2, \dots, \sigma_q) \in A_n^q : i^{\sigma_k} = j, i'^{\sigma_k} = j' \} | = \frac{(n-1)!n^{!q-1}}{2^q}.$$

Thus the assertion holds. ■

Lemma 4.6 Let $\mathcal{B}^{(k)} = \{B_{i,j}^{(k)}, i, j = 1, 2, \dots, n\}$, $k = 1, 2, \dots, q$. For any $\phi \in \text{Aut}(A\Gamma_n^q)$, There exists a $B_{i,j}^{(k)} \in \mathcal{B}^{(k)}$ such that $B_{i,j}^{(k)\phi} \in \mathcal{B}^{(k')}$ if and only if $B_{i,j}^{(k)\phi} \in \mathcal{B}^{(k')}$ for any $B_{i,j}^{(k)} \in \mathcal{B}^{(k)}$.

Proof. Suppose on the contrary that there exist two distinct $B_{i,j}^{(k)}, B_{i',j'}^{(k)} \in \mathcal{B}^{(k)}$ such that $B_{i,j}^{(k)\phi} \in \mathcal{B}^{(k')}$, $B_{i',j'}^{(k)\phi} \in \mathcal{B}^{(k'')}$ with $k' \neq k''$.

Since $B_{i,j}^{(k)} \neq B_{i',j'}^{(k)}$, we have $|B_{i,j}^{(k)} \cap B_{i',j'}^{(k)}| = 0$ or $\frac{(n-2)!n^{!q-1}}{2^q}$ by using Lemma 4.5 and its proof. So

$$\begin{aligned} |B_{i,j}^{(k)\phi} \cup B_{i',j'}^{(k)\phi}| &= |(B_{i,j}^{(k)} \cup B_{i',j'}^{(k)})^\phi| = |B_{i,j}^{(k)} \cup B_{i',j'}^{(k)}| \\ &= \frac{2(n-1)!n^{!q-1}}{2^q} \text{ or } \frac{2(n-1)!n^{!q-1} - (n-2)!n^{!q-1}}{2^q}. \end{aligned}$$

On the other hand,

$$\begin{aligned} B_{i,j}^{(k)\phi} \in \mathcal{B}^{(k')}, B_{i',j'}^{(k)\phi} \in \mathcal{B}^{(k'')} (k' \neq k'') &\Rightarrow |B_{i,j}^{(k)\phi} \cap B_{i',j'}^{(k)\phi}| = \frac{(n-1)!n^{!q-2}}{2^q} \\ &\Rightarrow |B_{i,j}^{(k)\phi} \cup B_{i',j'}^{(k)\phi}| = \frac{2(n-1)!n^{!q-1} - (n-1)!n^{!q-2}}{2^q}, \end{aligned}$$

which is a contradiction. Thus the assertion holds. ■

Lemma 4.7 Let $\mathcal{R}_i^{(k)} = \{B_{i,1}^{(k)}, B_{i,2}^{(k)}, \dots, B_{i,n}^{(k)}\}$ and $\mathcal{C}_j^{(k)} = \{B_{1,j}^{(k)}, B_{2,j}^{(k)}, \dots, B_{n,j}^{(k)}\}$, $k = 1, 2, \dots, q$. Then for any $x_1, x_2, \dots, x_n \in \mathcal{B}$, we have

$$x_1 \cup x_2 \cup \dots \cup x_n = A_n^q$$

if and only if there exist some $k \in \{1, 2, \dots, q\}$ and some i or $j \in \{1, 2, \dots, n\}$ such that $\{x_1, x_2, \dots, x_n\} = \mathcal{R}_i^{(k)}$ or $\mathcal{C}_j^{(k)}$.

Proof. Clearly if $\{x_1, x_2, \dots, x_n\} = \mathcal{R}_i^{(k)}$ or $C_j^{(k)}$ for some $k \in \{1, 2, \dots, q\}$ and some i or $j \in \{1, 2, \dots, n\}$, then $x_1 \cup x_2 \cup \dots \cup x_n = A_n^q$.

Assume that $x_1 \cup x_2 \cup \dots \cup x_n = A_n^q$. Since $\forall i, |x_i| = \frac{(n-1)!n^{q-1}}{2^q}$ and $|A_n^q| = \frac{n!q}{2^q}$, we have $x_i \cap x_j = \emptyset, \forall i, j, i \neq j$. Applying Lemma 4.5, we obtain $\{x_1, x_2, \dots, x_n\} = \mathcal{R}_i^{(k)}$ or $C_j^{(k)}$. ■

Lemma 4.8 *Let $\Omega = \{C_i^{(k)}, \mathcal{R}_j^{(k)}, i, j = 1, 2, \dots, n; k = 1, 2, \dots, q\}$. Then the action of $\text{Aut}(A\Gamma_n^q)$ on Ω can be induced by the action of $\text{Aut}(A\Gamma_n^q)$ on \mathcal{B} in Lemma 4.4, and it is faithful. Furthermore, any $\phi \in \text{Aut}(A\Gamma_n^q)$ is a permutation of Ω .*

Proof. First for any $\mathcal{R}_i^{(k)} \in \Omega$ and $\phi \in \text{Aut}(A\Gamma_n^q)$, we have

$$B_{i,1}^{(k)\phi} \cup B_{i,2}^{(k)\phi} \cup \dots \cup B_{i,n}^{(k)\phi} = (B_{i,1}^{(k)} \cup B_{i,2}^{(k)} \cup \dots \cup B_{i,n}^{(k)})^\phi = (A_n^q)^\phi = A_n^q.$$

So by Lemma 4.7, we have $\mathcal{R}_i^{(k)\phi} = \{B_{i,1}^{(k)\phi}, B_{i,2}^{(k)\phi}, \dots, B_{i,n}^{(k)\phi}\} \in \Omega$.

Similarly, for any $C_j^{(k)} \in \Omega$ and $\phi \in \text{Aut}(A\Gamma_n^q)$, we have $C_j^{(k)\phi} \in \Omega$.

Assume that $\phi \in \text{Aut}(A\Gamma_n^q)$ satisfies $\mathcal{R}_i^{(k)\phi} = \mathcal{R}_i^{(k)}$ and $C_j^{(k)\phi} = C_j^{(k)}$ for any $i, j \in \{1, 2, \dots, n\}$ and $k \in \{1, 2, \dots, q\}$. Then it suffices to show that ϕ is the identity map.

Since for any $B_{i,j}^{(k)} \in \mathcal{B}$, we have $\{B_{i,j}^{(k)}\} = (\mathcal{R}_i^{(k)} \cap C_j^{(k)})^\phi \subseteq \mathcal{R}_i^{(k)\phi} \cap C_j^{(k)\phi} = \mathcal{R}_i^{(k)} \cap C_j^{(k)} = \{B_{i,j}^{(k)}\}$. By Lemma 4.4, the action of $\text{Aut}(A\Gamma_n^q)$ on \mathcal{B} is faithful. Thus ϕ is the identity map.

For any $\omega_1, \omega_2 \in \Omega$ and $\phi \in \text{Aut}(A\Gamma_n^q)$,

$$\begin{aligned} \omega_1 \neq \omega_2 &\Leftrightarrow |\omega_1 \cup \omega_2| > n \\ &\Leftrightarrow |(\omega_1 \cup \omega_2)^\phi| > n \\ &\Leftrightarrow |\omega_1^\phi \cup \omega_2^\phi| > n \\ &\Leftrightarrow \omega_1^\phi \neq \omega_2^\phi. \end{aligned}$$

Thus ϕ is a permutation of Ω . ■

Lemma 4.9 *Let $\mathcal{R}^{(k)} = \{\mathcal{R}_1^{(k)}, \mathcal{R}_2^{(k)}, \dots, \mathcal{R}_n^{(k)}\}$, $C^{(k)} = \{C_1^{(k)}, C_2^{(k)}, \dots, C_n^{(k)}\}$ and $\Omega^{(k)} = \mathcal{R}^{(k)} \cup C^{(k)}$, $k = 1, 2, \dots, q$. For any $\phi \in \text{Aut}(A\Gamma_n^q)$, the following (i)-(iii) hold:*

(i) *There exists a $\sigma \in S_q$ such that $\Omega^{(k)\phi} = \Omega^{(k^\sigma)}$, $k = 1, 2, \dots, q$.*

(ii) *There exists some $\mathcal{R}_i^{(k)} \in \mathcal{R}^{(k)}$ such that $\mathcal{R}_i^{(k)\phi} \in \mathcal{R}^{(k')}$ if and only if $\mathcal{R}_i^{(k)} \in \mathcal{R}^{(k')}$ for any $\mathcal{R}_i^{(k)} \in \mathcal{R}^{(k)}$;*

(iii) *There exists some $\mathcal{R}_j^{(k)} \in \mathcal{R}^{(k)}$ such that $\mathcal{R}_j^{(k)\phi} \in C^{(k')}$ if and only if $\mathcal{R}_j^{(k)} \in C^{(k')}$ for any $\mathcal{R}_j^{(k)} \in \mathcal{R}^{(k)}$.*

Proof. (i) By Lemma 4.6, for any $k \in \{1, 2, \dots, q\}$ there exists a $l \in \{1, 2, \dots, q\}$ such that $\mathcal{B}^{(k)\phi} = \mathcal{B}^{(l)}$. Moreover, if $k \neq k'$, then by Lemma 4.4, we have $\mathcal{B}^{(k)\phi} \neq \mathcal{B}^{(k')\phi}$. Thus there exists a $\sigma \in S_q$ such that $\mathcal{B}^{(k)\phi} = \mathcal{B}^{(k^\sigma)}$, $k = 1, 2, \dots, q$. By Lemma 4.8, the assertion holds.

(ii) First by (i), there exists some $\mathcal{R}_i^{(k)} \in \mathcal{R}^{(k)}$ such that $\mathcal{R}_i^{(k)\phi} \in \Omega^{(k')}$ if and only if $\mathcal{R}_i^{(k)\phi} \in \Omega^{(k')}$ for any $\mathcal{R}_i^{(k)} \in \mathcal{R}^{(k)}$.

Suppose on the contrary that there exist $i, j (\neq i) \in \{1, 2, \dots, n\}$ such that $\mathcal{R}_i^{(k)\phi} \in \mathcal{R}^{(k')}$ and $\mathcal{R}_j^{(k)\phi} \in C^{(k')}$.

Note that

$$\begin{aligned}\mathcal{R}_i^{(k)} \cap \mathcal{R}_j^{(k)} &= \emptyset \text{ for } i \neq j, \\ \mathcal{R}_i^{(k)} \cap C_j^{(k)} &= \{B_{i,j}^{(k)}\} \text{ for any } i, j.\end{aligned}$$

Then

$$i \neq j \Rightarrow \mathcal{R}_i^{(k)} \cap \mathcal{R}_j^{(k)} = \emptyset \Rightarrow |\mathcal{R}_i^{(k)} \cup \mathcal{R}_j^{(k)}| = 2n \Rightarrow |\mathcal{R}_i^{(k)\phi} \cup \mathcal{R}_j^{(k)\phi}| = |(\mathcal{R}_i^{(k)} \cup \mathcal{R}_j^{(k)})^\phi| = 2n.$$

On the other hand,

$$\mathcal{R}_i^{(k)\phi} \in \mathcal{R}^{(k')}, \mathcal{R}_j^{(k)\phi} \in C^{(k')} \Rightarrow |\mathcal{R}_i^{(k)\phi} \cap \mathcal{R}_j^{(k)\phi}| = 1 \Rightarrow |\mathcal{R}_i^{(k)\phi} \cup \mathcal{R}_j^{(k)\phi}| = 2n - 1,$$

which is a contradiction. Thus the assertion holds.

(iii) The proof of (iii) is similar to that of (ii). ■

Lemma 4.10 For $n \geq 5$, we have

$$|\text{Aut}(A\Gamma_n^q)| \leq q!n!^{2q}.$$

Proof. By (i) of Lemma 4.9, for any $\phi \in \text{Aut}(A\Gamma_n^q)$, there exists a $\sigma \in S_q$ such that $\Omega^{(k)\phi} = \Omega^{(k^\sigma)}$ ($k = 1, 2, \dots, q$). Using (ii) and (iii) of Lemma 4.9 we obtain the following disjoint alternatives:

- (i) $\mathcal{R}^{(k)\phi} = \mathcal{R}^{(k^\sigma)}$ and $C^{(k)\phi} = C^{(k^\sigma)}$;
- (ii) $\mathcal{R}^{(k)\phi} = C^{(k^\sigma)}$ and $C^{(k)\phi} = \mathcal{R}^{(k^\sigma)}$.

So $\text{Aut}(A\Gamma_n^q) = \bigcup_{\sigma \in S_q} \{\phi \in \text{Aut}(A\Gamma_n^q) : \Omega^{(k)\phi} = \Omega^{(k^\sigma)}, k = 1, 2, \dots, q\}$. Hence, if we can prove the last two inequalities, then we have

$$\begin{aligned}|\text{Aut}(A\Gamma_n^q)| &\leq \sum_{\sigma \in S_q} |\{\phi \in \text{Aut}(A\Gamma_n^q) : \Omega^{(k)\phi} = \Omega^{(k^\sigma)}, k = 1, 2, \dots, q\}| \\ &\leq \sum_{\sigma \in S_q} \prod_{k=1}^q (|\{\phi \in \text{Aut}(A\Gamma_n^q) : \mathcal{R}^{(k)\phi} = \mathcal{R}^{(k^\sigma)}, C^{(k)\phi} = C^{(k^\sigma)}\}| + \\ &\quad |\{\phi \in \text{Aut}(A\Gamma_n^q) : \mathcal{R}^{(k)\phi} = C^{(k^\sigma)}, C^{(k)\phi} = \mathcal{R}^{(k^\sigma)}\}|) \\ &\leq \sum_{\sigma \in S_q} \prod_{k=1}^q \left(\frac{n!^2}{2} + \frac{n!^2}{2}\right) \\ &= \sum_{\sigma \in S_q} n!^{2q} \\ &= q!n!^{2q}.\end{aligned}$$

Now we show that

$$|\{\phi \in \text{Aut}(A\Gamma_n^q) : \mathcal{R}^{(k)\phi} = \mathcal{R}^{(k^\sigma)}, C^{(k)\phi} = C^{(k^\sigma)}\}| \leq \frac{n!^2}{2},$$

$$|\{\phi \in \text{Aut}(A\Gamma_n^q) : \mathcal{R}^{(k)\phi} = C^{(k^\sigma)}, C^{(k)\phi} = \mathcal{R}^{(k^\sigma)}\}| \leq \frac{n!^2}{2}.$$

Indeed, for any $\phi \in \text{Aut}(A\Gamma_n^q)$ such that $\mathcal{R}^{(k)\phi} = \mathcal{R}^{(k^\sigma)}$, $C^{(k)\phi} = C^{(k^\sigma)}$, define $\phi_1, \phi_2 \in S_n$ as $\mathcal{R}_i^{(k)\phi} = \mathcal{R}_{i\phi_1}^{(k^\sigma)}$, $C_j^{(k)\phi} = C_{j\phi_2}^{(k^\sigma)}$.

Since $\{B_{ij}^{(k)\phi}\} = (\mathcal{R}_i^{(k)} \cap C_j^{(k)})^\phi \subseteq \mathcal{R}_i^{(k)\phi} \cap C_j^{(k)\phi} = \mathcal{R}_{i\phi_1}^{(k^\sigma)} \cap C_{j\phi_2}^{(k^\sigma)} = \{B_{i\phi_1 j\phi_2}^{(k^\sigma)}\}$, we have

$$\{(1, 1, \dots, 1)^\phi\} \in \left(\bigcap_{i=1}^n B_{ii}^{(k)\phi}\right)^\phi \subseteq \bigcap_{i=1}^n B_{ii}^{(k)\phi} = \bigcap_{i=1}^n B_{i\phi_1 i\phi_2}^{(k^\sigma)} = \{(\tau_1, \tau_2, \dots, \tau_q) \in A_n^q : \tau_{k^\sigma} = \phi_1^{-1}\phi_2\}.$$

So $(1, 1, \dots, 1)^\phi = (\tau_1, \dots, \tau_{k^\sigma-1}, \phi_1^{-1}\phi_2, \tau_{k^\sigma+1}, \dots, \tau_q) \in A_n^q$, which implies that $\phi_1^{-1}\phi_2 \in A_n$.

Thus

$$\begin{aligned} & |\{\phi \in \text{Aut}(A\Gamma_n^q) : \mathcal{R}^{(k)\phi} = \mathcal{R}^{(k^\sigma)}, C^{(k)\phi} = C^{(k^\sigma)}\}| \\ &= |\{\phi \in \text{Aut}(A\Gamma_n^q) : \mathcal{R}^{(k)\phi} = \mathcal{R}^{(k^\sigma)}, C^{(k)\phi} = C^{(k^\sigma)}, \phi_1^{-1}\phi_2 \in A_n\}| \\ &\leq \frac{n!^2}{2}. \end{aligned}$$

Similarly,

$$|\{\phi \in \text{Aut}(A\Gamma_n^q) : C^{(k)\phi} = \mathcal{R}^{(k^\sigma)}, \mathcal{R}^{(k)\phi} = C^{(k^\sigma)}\}| \leq \frac{n!^2}{2}.$$

Thus the assertion holds. ■

Proof. (of Theorem 1.3). By Lemma 4.3, we have $|\text{Aut}(A\Gamma_n^q)| \geq q!n!^{2q}$. On the other hand, by Lemma 4.10, we obtain $|\text{Aut}(A\Gamma_n^q)| \leq q!n!^{2q}$. Hence $|\text{Aut}(A\Gamma_n^q)| = q!n!^{2q}$, and by Lemma 4.3 again, the assertion holds. ■

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